Due Sun

2.3 – Properties of Determinants; Cramer's Rule

Properties of Determinants – let A be an $n \times n$ matrix

- $\det(kA) = k^n \det(A)$ – one factor of k for each row

Note: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow \det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$

$\begin{vmatrix} 3 & 6 \\ 3 & 4 \end{vmatrix} = -6 = 3(-2)$ As in 2.2

But $3A = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} \Rightarrow \det(3A) = \begin{vmatrix} 3 & 6 \\ 9 & 12 \end{vmatrix}$

$= -18 = 3^2(-2)$

Again note that $3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ means something different from $3\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

- **[Lemma 2.3.2** (prelude to a later result)
  If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then $\det(EB) = \det(E)\det(B)$.] (In brackets because it's here to support the next point)

Suppose E results from multiplying the i^{th} row of I by k.

Then EB is

$$EB = \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vdots \\ k\vec{e}_i \\ \vdots \\ \vec{e}_n \end{bmatrix} B = \begin{bmatrix} \vec{e}_1 B \\ \vec{e}_2 B \\ \vdots \\ k\vec{e}_i B \\ \vdots \\ \vec{e}_n B \end{bmatrix}$$

So the i^{th} row of EB is the i^{th} row of B multiplied by k.

By Thm 2.2.3 a, $\det(E) = k$, so
$$\det(EB) = k\det(B) = \det(E)\det(B).$$

- **Theorem 2.3.4** If A and B are square matrices of the same size, then $\det(AB) = \det(A)\det(B)$.

Not a formal proof

If A and/or B is not invertible, then AB is not invertible (see thm 2.3.3 below). Then $0 = 0$. If A & B are invertible, we use products of elementary matrices and Lemma 2.3.2.

- **Theorem 2.3.5** If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

Pf: Let A be an $n \times n$ invertible matrix.

Then $\exists$ A^{-1} $\ni$ $A^{-1}A = I$

there is, such that

$$\det(A^{-1}A) = \det I$$

$$\det(A^{-1})\det(A) = 1 \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

Since $\det(A) \neq 0$.

Theorem 2.3.3 A square matrix A is invertible if and only if $\det(A) \neq 0$.

Theorem 2.3.8 Equivalent Statements (extends Theorem 1.6.4)

If A is an $n \times n$ matrix, then the following statements are equivalent.

a) A is invertible.

b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

c) The reduced row echelon form of A is I_n.

d) A is expressible as a product of elementary matrices.

e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

g) $\det(A) \neq 0$.

$\Rightarrow$ A invertible $\Rightarrow$ A is a product of elementary matrices $\Rightarrow \det(A) \neq 0$.

$\det(A) \neq 0 \Rightarrow \operatorname{rref}(A)$ does not have a row of zeros (Thm 2.2.1)

$\Rightarrow \operatorname{rref}(A) = I \Rightarrow A$ is invertible.

16. Find the values of k for which the matrix A is invertible.

$A = \begin{bmatrix} k & 2 \\ 2 & k \end{bmatrix}$

$\det(A) \neq 0$

$\begin{vmatrix} k & 2 \\ 2 & k \end{vmatrix} = k^2 - 4 = 0$

$\Rightarrow \{k \in \mathbb{R} \mid k \neq \pm 2\}$

35. In each part, find the determinant given that A is a 3×3 matrix for which $\det(A) = 7$.

a. $\det(3A)$

b. $\det(A^{-1})$

c. $\det(2A^{-1})$

d. $\det((2A)^{-1})$

a) $3^3 \cdot 7 = 189$

b) $\frac{1}{\det(A)} = \frac{1}{7}$

c) $2^3\left(\frac{1}{7}\right) = \frac{8}{7}$

d) $\frac{1}{2^3 \cdot 7} = \frac{1}{56}$

Definition 1: If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the **matrix of cofactors from A**. The transpose of this matrix is called the **adjoint of A** and is denoted by $\mathrm{adj}(A)$.

Theorem 2.3.6 Inverse of a Matrix Using Its Adjoint

If A is an invertible matrix, then $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$.

20. Decide whether the matrix is invertible, and if so, use the adjoint to find its inverse.

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$$

$$\det(A) = 2(-12) + 0(-4) + 3(6) = -6 \quad A \text{ is invertible } \checkmark$$

$$M_{11} = \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} = -12 \qquad M_{12} = \begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix} = 4 \qquad M_{13} = \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix} = 6$$

$C_{11} = -12$ $\quad C_{12} = -4$ $\quad C_{13} = 6$

$M_{21} = 0$ $\quad M_{22} = -2$ $\quad M_{23} = 0$

$C_{21} = 0$ $\quad C_{22}\ -2$ $\quad C_{23} = 0$

$M_{31} = -9$ $\quad M_{32} = 4$ $\quad M_{33} = 6$

$C_{31} = -9$ $\quad C_{32} = -4$ $\quad C_{33} = 6$

$$[C_{ij}] = \begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix} \quad \operatorname{adj}(A) = [C_{ij}]^T = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = -\frac{1}{6} \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

$$\boxed{A^{-1} = \begin{bmatrix} 2 & 0 & 3/2 \\ 2/3 & 1/3 & 2/3 \\ -1 & 0 & -1 \end{bmatrix}}$$

2×2: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad [C_{ij}] = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ $\quad \operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Theorem 2.3.7 Cramer's Rule

If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \qquad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, \qquad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

26. Solve by Cramer's rule.

$$\begin{aligned} x - 4y + z &= 6 \\ 4x - y + 2z &= -1 \\ 2x + 2y - 3z &= -20 \end{aligned}$$

$A = \begin{bmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{bmatrix}$

$-2 + 4 + 48 = 50$

$\det(A) = \begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} \begin{matrix} 1 & -4 \\ 4 & -1 \\ 2 & 2 \end{matrix}$ $\qquad -5 - 50 = -55$

$3 - 16 + 8 = -5$

$\det(A_1) = \begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix} = 144$

$\det(A_2) = \begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix} = 61$

$$\det(A_3) = \begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix} = -230$$

$$x = \frac{\det(A_1)}{\det(A)} = -\frac{144}{55}$$

$$y = \frac{\det(A_2)}{\det(A)} = -\frac{61}{55}$$

$$z = \frac{\det(A_3)}{\det(A)} = \frac{230}{55} = \frac{46}{11}$$

$$\boxed{(x, y, z) = \left(-\frac{144}{55}, -\frac{61}{55}, \frac{46}{11}\right)}$$